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VIBRATIONS OF AN ELASTIC INHOMOGENEOUS SOLID WEAKENED BY A CIRCULAR SLIT

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The vibrations of an elastic homogeneous solid, weakened by a circular slit, were discussed in [1]. A solution of the corresponding static problem was set forth in [2, 3]. For a medium whose Lamé parameters and density depend on the coordinate z, the analogous problem is complicated considerably and admits of an effective analytical solution only for certain cases of the dependence of the above functions on z and of fixed values of the Poisson coefficient.

The present article discusses the static and dynamic problems of determining the displacement in an inhomogeneous elastic solid weakened by a circular slit.

§1. We consider a solid elastic medium, occupying the whole space. The Lamé parameters λ and μ and the density of the medium ρ depend on z:

$$\mu = \mu_0(a|z| + 1)^{3-4\nu}, \quad \rho = \rho_0(a|z| + 1)^{4(1-2\nu)}, \tag{1.1}$$

where ν is the Poisson coefficient, assumed constant. As is shown in [4], the equations of motion of such a medium in the case of axial symmetry, in a cylindrical system of coordinates, can be written in the form

$$\begin{gathered}\nabla^2\Phi - v_1^{-2}\varepsilon^{1-4\nu}\frac{\partial^2\Phi}{\partial t^2} = 0,\\ \nabla^2\psi - v_2^{-2}\varepsilon^{1-4\nu}\frac{\partial^2\psi}{\partial t^2} = 0,\end{gathered} \tag{1.2}$$

where $\nabla^2 = \partial^2/\partial r^2 + \partial/r\partial r + \partial^2/\partial z^2$; v_1 and v_2 are the velocities of the deformation waves for $z = 0$, $\varepsilon = a|z| + 1$. The functions Φ and ψ are connected with the vector displacement $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2 = u_r\mathbf{i}_r + u_z\mathbf{i}_z$ by the dependences

$$\mathbf{u}_1\varepsilon^{2(1-2\nu)} = \nabla\Phi, \quad \mathbf{u}_2\varepsilon^{2(1-2\nu)} = \nabla \times (\mathbf{i}_\varphi\partial\psi/\partial r), \tag{1.3}$$

where $\mathbf{i}_\varphi$ is a unit vector.

In the plane $z = 0$ there is a circular opening of radius $r = 1$ with its center at the origin of coordinates. It is required to solve the system of equations (1.2) under the assumption that the displacements and stresses in the vicinity of the slit are the same as in a semiinfinite body $z \geq 0$, where, at the free surface $z = 0$, the following boundary conditions obtain:

$$\begin{gathered}\sigma_z = -p_s - p_0\exp(-i\omega t), \quad 0 \leqslant r < 1,\\ \tau_{rz} = 0, \quad 0 \leqslant r < \infty, \quad u_z = 0, \quad r > 1,\end{gathered} \tag{1.4}$$

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where σ_z, and τ_{rz} are components of the tensor of the stresses; u_z is the axial displacement; p_s is the static pressure, excluding the possibility of closing of the slit.

We solve Eqs. (1.2) under the condition that $\nu = 1/4$. Since the stresses are even with respect to z, the problem is considered for the half-space $z \geq 0$. The solution of Eqs. (1.2), bounded for $z \geq 0$, is sought in the form

$$\Phi = \int_0^\infty A_1(\alpha) J_0(\alpha r) \alpha e^{-\delta z} d\alpha,$$
$$\psi = \int_0^\infty A_2(\alpha) J_0(\alpha r) \alpha e^{-\eta z} d\alpha, \tag{1.5}$$

where $\delta = \sqrt{\alpha^2 - k^2/3}$; $\eta = \sqrt{\alpha^2 k^2}$; $k^2 = \omega^2/v_2^2$; α is the separation parameter of Eqs. (1.2); $J_0(\alpha r)$ is a Bessel function. We select the signs of the radicals in accordance with the conditions

$$\delta = (\alpha^2 - k^2/3)^{1/2},\ \alpha > k/\sqrt{3},$$
$$\eta = (\alpha^2 - k^2)^{1/2},\ \alpha > k,$$
$$\delta = -i(k^2/3 - \alpha^2)^{1/2},\ 0 \leqslant \alpha < k/\sqrt{3},\ \eta = -i(k^2 - \alpha^2)^{1/2}, \tag{1.6}$$
$$0 \leqslant \alpha < k.$$

Using dependences (1.3), we express the stress and the displacement in terms of the functions Φ and ψ:

$$\sigma_z = \mu\left[\left(\nabla^2 + 2\frac{\partial^2}{\partial z^2} - \frac{3a\partial}{\varepsilon\partial z}\right)\Phi + R\left(2\frac{\partial}{\partial z} - \frac{3a}{\varepsilon}\right)\psi\right],$$
$$\tau_{rz} = \mu\frac{\partial}{\partial r}\left[\left(2\frac{\partial}{\partial z} - a\varepsilon^{-1}\right)\Phi + \left(\nabla^2 - 2\frac{\partial^2}{\partial z^2} + \frac{a\partial}{\varepsilon\partial z}\right)\psi\right],$$
$$u_z = \varepsilon^{-1}\left(\frac{\partial\Phi}{\partial z} + R\psi\right),\ u_r = \varepsilon^{-1}\frac{\partial}{\partial r}\left(\Phi - \frac{\partial\psi}{\partial z}\right), \tag{1.7}$$
$$R = \frac{\partial}{r\partial r}\left(r\frac{\partial}{\partial r}\right).$$

Using (1.4) and the second relationship of (1.7), we find

$$A_1(\alpha) = -(\alpha^2 + \eta^2 + a\eta)(2\delta + a)^{-1}A_2(\alpha). \tag{1.8}$$

We take the first boundary condition in the form

$$\sigma_z = -p_0 e^{-i\omega t},$$

since the static stress can be taken into consideration independently of the solution of the dynamic problem. Using the other boundary conditions (1.4) and (1.8), we obtain

$$\mu\int_0^\infty [\alpha \mathrm{Ra}(\alpha) A_2(\alpha) J_0(\alpha r)](2\delta + a)^{-1} d\alpha = -p_0,\ r < 1,$$
$$\int_0^\infty \alpha[\delta k^2 + a(\alpha^2 - \eta\delta)] A_2(\alpha) J_0(\alpha r)(2\delta + a)^{-1} d\alpha = 0,\ r > 1,$$

where $\mathrm{Ra}(\alpha)$ is the Rayleigh function of a medium with the properties (1.1):

$$\mathrm{Ra}(\alpha) = (2\alpha^2 - k^2)^2 - 4\alpha^2\delta\eta - k^2 a(3\delta + \eta) + 3a^2(\delta\eta - \alpha^2).$$

Denoting

$$D(\alpha) = \frac{[\delta k^2 + a(\alpha^2 - \delta\eta)]\alpha A_2(\alpha)}{2\delta + a};$$
$$H(\alpha) = \frac{3\mathrm{Ra}(\alpha)}{4\alpha[\delta k^2 + a(\alpha^2 - \delta\eta)]} - 1, \tag{1.9}$$

we arrive at the pairwise integral equations

$$\int_0^\infty D(\alpha) J_0(\alpha r) d\alpha = 0,\ r > 1,$$
$$\int_0^\infty \alpha[1 + H(\alpha)] D(\alpha) J_0(\alpha r) d\alpha = -\frac{3p_0}{4\mu_0},\ r < 1. \tag{1.10}$$

Equations (1.10) are a special case of the more general system

$$\int_0^\infty \alpha^\gamma [1 + H(\alpha)] D(\alpha) J_\varkappa (\alpha r)\, d\alpha = f(r),\ r < d,$$

$$\int_0^\infty D(\alpha) J_\varkappa (\alpha r)\, d\alpha = 0,\ r > d.$$

In our case, $\gamma = 1$, $\varkappa = 0$, $d = 1$, $f(r) = -3p_0/4\mu_0$. In accordance with [5], we seek the solution in the form

$$D(\alpha) = (2\alpha)^{1-\frac{\gamma}{2}} \Gamma^{-1}\left(\frac{\gamma}{2}\right) \int_0^d \xi^{\frac{1}{2}} \theta(\xi) J_{\varkappa+\frac{\gamma}{2}} (\alpha\xi)\, d\xi, \tag{1.11}$$

where $\theta(\xi)$ satisfies a Fredholm equation of the second kind:

$$\theta(x) + \frac{1}{\pi} \int_0^d M(x, \xi)\, \theta(\xi)\, d\xi = F(x); \tag{1.12}$$

$$M(x, \xi) = \pi (x, \xi)^{\frac{1}{2}} \int_0^\infty \alpha H(\alpha) J_{\varkappa+\frac{1}{2}\gamma} (\alpha x) J_{\varkappa+\frac{1}{2}\gamma} (\alpha\xi)\, d\alpha, \tag{1.13}$$

and $\Gamma(\gamma)$ is a gamma function.

If $0 < \gamma < 2$, then

$$F(x) = x^{-\varkappa-\frac{1}{2}\gamma+\frac{1}{2}} \int_0^d f(r) r^{\varkappa+1} (x^2 - r^2)^{-1+\frac{1}{2}\gamma} dr. \tag{1.14}$$

Taking account of the values of γ, $\varkappa$, and d, from (1.11)-(1.14) we find

$$D(\alpha) = \frac{2}{\pi} \int_0^1 \theta(\xi) \sin(\alpha\xi)\, d\xi,\quad F(x) = -\frac{3p_0 x}{4\mu_0}; \tag{1.15}$$

$$M(x, \xi) = \pi (x\xi)^{\frac{1}{2}} \int_0^\infty \alpha H(\alpha) J_{\frac{1}{2}} (\alpha x) J_{\frac{1}{2}} (\alpha\xi)\, d\alpha. \tag{1.16}$$

To go over to the static problem, we find the limit of $H(\alpha, \omega)$ when ω tends to zero:

$$\lim_{\omega\to 0} H(\alpha, \omega) = \frac{3(2\alpha + 3a)^2}{4\alpha(3\alpha + 2a)} - 1.$$

Then the kernel (1.16) can be represented in the form

$$M(x, \xi) = \int_0^\infty \frac{(10\alpha + 27a)a}{4\alpha(3\alpha + 2a)} \sin(\alpha\xi) \sin(\alpha x)\, d\alpha = \frac{a}{48}\left[81 \ln \frac{\xi + x}{\xi - x} + \right.$$

$$\left. + 61 (\sin\theta_1 \operatorname{si}\theta_1 + \cos\theta_1 \operatorname{ci}\theta_1 - \sin\theta_2 \operatorname{si}\theta_2 - \cos\theta_2 \operatorname{ci}\theta_2)\right],$$

$$\theta_1 = \frac{2}{3} a(\xi - x),\ \theta_2 = \frac{2}{3} a(\xi + x). \tag{1.17}$$

Thus, the static problem is reduced to the solution of the integral equation (1.12) with the kernel (1.17), in which siθ and ciθ are the integral sine and cosine. As can be seen from (1.17), at the diagonal $\xi = x$, the kernel has a logarithmic singularity. All the Fredholm theorems [6] are applicable to equations with such kernels. Since the logarithmic singularity has an order < 1/2, the kernel **(1.17) satisfies the condition**

$$\int_0^1 |M(x, \xi)|^2 d\xi \leqslant A, \tag{1.18}$$

where A is some finite constant. Since the integration integral is finite, then from (1.18) there follows the satisfaction of the second equation:

$$\int_0^1 \int_0^1 |M(x, \xi)|^2 dx d\xi = B^2 \leqslant A. \tag{1.19}$$

It can be shown that the parameter of Eq. (1.12) with the kernel (1.17) $\chi = 61a/48\pi$ is not an eigenvalue of the equation. Then, on the basis of the Fredholm alternative, it has a singular solution. Since the right-hand side of Eq. (1.2) is a bounded function, the solution of the equation has the same property.

For an approximate search for the solution, we use the method of successive approximations:

$$\theta(x) = \theta_0(x) + \sum_{n=1}^{\infty} \chi^n \theta_n(x) = F(x) + \sum_{n=1}^{\infty} \chi^n \int_0^1 K_n(x, \xi) F(\xi) d\xi, \tag{1.20}$$

where $K_n(x, \xi)$ is an iterated kernel of the n-th order.

As is well known [6], if the kernel of Eq. (1.12) satisfies the conditions (1.18), (1.19) and

$$\chi < B^{-1}, \tag{1.21}$$

then the series (1.20) converges absolutely and uniformly. It can be shown that, for weakly inhomogeneous media, the condition (1.21) is satisfied. The zero approximation of the solution (1.20) corresponds to the case of a homogeneous medium ($a = 0$):

$$\theta_0 = -\frac{3}{4\mu_0} p_0 x. \tag{1.22}$$

Carrying out the necessary computations, we find the following term of the solution (1.20):

$$\theta_1(x) = \frac{a}{24\pi}\{81(x\ln x - x) + \frac{61}{2}[(\cos\varphi_1 \operatorname{si}\varphi_1 - \cos\varphi_2 \operatorname{si}\varphi_2 + \sin\varphi_2 \operatorname{ci}\varphi_2 -$$

$$-\sin\varphi_1 \operatorname{ci}\varphi_1)\frac{3}{2a} - (\sin\varphi_1 \operatorname{si}\varphi_1 - \sin\varphi_2 \operatorname{si}\varphi_2 - \cos\varphi_2 \operatorname{ci}\varphi_2 + \ln\varphi_2 + \cos\varphi_1 \operatorname{ci}\varphi_1 - \ln\varphi_1)\frac{9}{4a^2}]\}\frac{3p_0}{4\mu_0},$$

$$\varphi_1 = (x+1)\frac{2}{3}a, \quad \varphi_2 = (x-1)\frac{2}{3}a. \tag{1.23}$$

Acting analogously, we can also find the other terms. They are all bounded in the interval (0, 1) and, in the case of a homogeneous medium ($a = 0$), revert to zero.

§ 2. Let us calculate the axial and radial displacements in the half-space $z \geq 0$. Using relationships (1.5), (1.7)-(1.9), and (1.15), we obtain

$$u_z = \int_0^{\infty} \frac{[\delta(2\alpha^2 - k^2 + a\eta) e^{-\delta z} - \alpha^2(2\delta + a) e^{-z\eta}] D(\alpha) J_0(\alpha r) d\alpha}{\delta k^2 + a(\alpha^2 - \eta\delta)}. \tag{2.1}$$

Letting ω approach zero, and using the L'Hospital's rule, we arrive at an expression for the static axial displacement:

$$u_{zs} = \frac{3(p_s + p_0)}{2\mu_0\pi} \int_0^{\infty}\int_0^1 \theta'(\xi, a) \sin(\alpha\xi) \left(1 + \frac{2z\alpha}{3} - \frac{az\alpha}{3(3\alpha + 2a)}\right) e^{-z\alpha} J_0(\alpha r) d\alpha.$$

Taking as $\theta = (3p_0/4\mu_0)\theta'$ the sum of the two terms (1.22) and (1.23), we obtain a first approximation for the axial displacement in an inhomogeneous medium:

$$u_{zs} = \frac{3(p_0 + p_s)}{2\pi\mu_0} \int_0^{\infty} \left[\frac{\sin\alpha}{\alpha^2} - \frac{\cos\alpha}{\alpha} + \int_0^1 \theta_1'(\xi, a) \sin(\alpha\xi) d\xi\right]\left[1 + \frac{2z\alpha}{3} - \frac{az\alpha}{3(3\alpha + 2a)}\right] e^{-z\alpha} J_0(\alpha r) d\alpha. \tag{2.2}$$

In the case of a homogeneous medium, $\theta'_1(\xi, a) = a = 0$, and expression (2.2) coincides with the result obtained in [2]. Using (1.7) and the other necessary dependences, we find the radial displacement,

$$u_r = -\int_0^{\infty} \frac{\alpha D(\alpha)[-(\alpha^2 + \eta^2 + a\eta) e^{-\delta z} + \eta(2\delta + a) e^{-z\eta}]}{\delta k^2 + a(\alpha^2 - \eta\delta)} J_1(\alpha r) d\alpha. \tag{2.3}$$

The corresponding static displacement is brought into the form

$$u_{rs} = \frac{3(p_s + p_0)}{2\pi\mu_0} \int_0^{\infty} \left[\frac{\sin\alpha}{\alpha^2} - \frac{\cos\alpha}{\alpha} + \int_0^1 \theta_1'(\xi, a) \sin(\alpha\xi) d\xi\right]\left[\frac{-1 + 2z\alpha}{3} + \frac{a(1 - 7z\alpha - 3z)}{3(6\alpha + 4a)}\right] e^{-z\alpha} J_1(\alpha r) d\alpha. \tag{2.4}$$

In the case of a homogeneous medium, (2.4) also coincides with the result obtained by Sneddon [2]. Formulas (2.2) and (2.4) make it possible to calculate the first correction to the true solution for a homogeneous medium, due to the inhomogeneities of the latter.

For an approximate solution of the dynamic problem, it is useful to represent the kernel (1.16) in another form. The corresponding procedure of the calculations is set forth in [1]; therefore, we give here only the final result. Integrating in the complex plane, and selecting the signs of the radicals δ and η, in accordance with (1.6) we obtain

$$M(x,\ \xi) = k\sum_{q=1}^{2}\int_{a_q}^{b_q} \Phi_q(\zeta,\ \beta)\, e^{ixk\zeta} \sin(\xi_k \zeta)\, d\zeta,$$

where

$$\Phi_1 = \frac{\delta_1[(2\zeta^2-1)^2 + 4\zeta^2\delta_1\eta_1] + \beta^2(\zeta^2+\delta_1\eta_1)\eta_1}{\delta_1^2 + \beta^2(\zeta^2+\delta_1\eta_1)^2}; \tag{2.5}$$

$$\Phi_2 = \frac{\left[4\delta_2^2 + \beta(4\delta_2+\beta)\right]\zeta^2}{(\delta_2+\beta\zeta^2)^2 + \beta^2\delta_2^2\eta_1^2},$$

$$a_1 = 0,\ a_2 = b_1 = 1/\sqrt{3},\ b_2 = 1, \qquad \zeta = \alpha/k,\ \delta_1 = \sqrt{1/3-\zeta^2},\ \eta_1 = \sqrt{1-\zeta^2},\ \delta_2 = \sqrt{\zeta^2 - 1/3},\ \beta = a/k. \tag{2.6}$$

Postulating that $0 \le a < k < 1$, we expand the kernel (2.5) in a Laurent series and a Maclaurin series in terms of the parameters k and a:

$$M(x,\ \xi) = \sum_{n=1}^{\infty}\sum_{m=1}^{n} \frac{k^{m+1}\, i^{m-1}\, \beta^{n-m}}{m!\,(n-m)!} M_m(x,\ \xi)\, E_{m,n-m}, \tag{2.7}$$

$$E_{m,n-m} = \frac{1}{(n-m)!}\sum_{q=1}^{2}\int_{a_q}^{b_q}\left[\left(\frac{\partial^{n-m}\Phi_q}{\partial\beta^{n-m}}\right)_{\beta=0}\right]\zeta^m\, d\zeta; \qquad M_m = \frac{1}{2}\left[(x+\xi)^m + (|x-\xi|)^m\right]. \tag{2.8}$$

The possibility of expanding the functions (2.6) in terms of the parameter β comes from the following considerations. For the parameter β we shall allow not only real, but also complex values (a medium with the absorption of energy). Then, with a fixed value of ζ, the functions Φ_q can be regarded as analytical functions of the variable β, each of which has two poles. A zero value of the parameter β is a regular point. Therefore, in a small circle with its center at the point $\beta = 0$, both functions are represented by a Taylor series. Since the coefficients of this series are integrable functions of the variable ζ, after their integration we obtain a converging series with the coefficients (2.8). Out of this there follows also the convergence of a series in powers of the real parameter β.

In accordance with the expansion (2.7), we seek the solution of Eq. (1.12) in the form

$$\theta(x) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} \theta_{m,n-m}(x)\, k^m \beta^{n-m} = \theta_{00}(x) + k\theta_{10}(x) + \beta\theta_{01}(x) + k^2\theta_{20}(x) + \cdots \tag{2.9}$$

Substituting (2.7) and (2.9) into (1.12) and equating terms with identical powers of the parameters, we obtain

$$\theta_{m+1,\ n-m} = -\frac{1}{\pi}\sum_{\gamma=0}^{n-m}\sum_{q=0}^{m-1} \frac{i^{m-q} E_{m-q,\ n-m-\gamma}}{(m-q)!\,(n-m-\gamma)!}\int_0^1 \theta_{q,\gamma}(\xi)\, M_{m-q}(x,\ \xi)\, d\xi. \tag{2.10}$$

For θ_{00}, we obtain directly a value equal to $3p_0x/4\mu_0$. In constructing formula (2.10), it was taken into consideration that, as follows from (1.12), all $\theta_{0\gamma}$ and $\theta_{1\gamma}$ ($\gamma = 1, 2, 3, \ldots$) are equal to zero. We write in explicit form several terms of the sought solution:

$$\begin{aligned}\theta(x) = \frac{3}{4\mu_0} p_0 \Bigg\{ &-x + \frac{1}{2}\left(x - \frac{x^3}{3}\right) k^2 \sum_{n=0}^{\infty}\frac{E_{1n}}{n!}\beta^n + \frac{2x}{3\pi}k^3 i \sum_{n=0}^{\infty}\frac{E_{2n}}{n!}\beta^n + \\ &+ k^4\left[\frac{x}{4\pi}\left(3E_{30} - \frac{5E_{10}^2}{6\pi}\right) + \frac{x^3}{2\pi}\left(E_{30} + \frac{E_{10}^2}{6\pi} - \frac{x^5}{20\pi}\left(E_{30} + \frac{E_{10}^2}{6\pi}\right)\right]\right. + \\ &+ k^4\beta\left[\frac{x}{4\pi}\left(3E_{31} - \frac{5E_{10}E_{11}}{6\pi}\right) + \frac{x^3}{2\pi}\left(1 - \frac{x^2}{10}\right)\left(E_{31} + \frac{E_{10}E_{11}}{\pi}\right) + \cdots\right. \\ &+ k^5 i\left[\frac{x}{5\pi}\left(4E_{40} - \frac{3E_{10}E_{20}}{\pi}\right) + \frac{x^3}{3\pi}\left(4E_{40} + \frac{E_{10}E_{20}}{3\pi}\right)\right] + k^5\beta i\left[\frac{x}{5\pi}\left(4E_{41} - \right.\right. \\ &\left.\left. - \frac{3}{\pi}(E_{11}E_{20} + E_{10}E_{21})\right) + \frac{x^3}{3\pi}\left(4E_{41} + \frac{1}{3\pi}(E_{11}E_{20} + E_{10}E_{21})\right)\right] + \cdots \Bigg\}.\end{aligned}$$

In the case of a homogeneous medium ($\beta = 0$), the solution obtained coincides with the solution of [1].

The axial and radial displacements in the half-space are determined by formulas (2.1) and (2.3). We find the axial displacement in the plane of the slit z = 0. Substituting z = 0 into (2.1), and replacing $D(\alpha)$ by its value from (1.16), we obtain

$$u_{z0} = -\frac{3}{2\pi\mu_0}\int_0^\infty \left\{\int_0^1 \theta'(\xi)\sin(\alpha\xi)\,d\xi\right\} J_0(\alpha r)\,d\alpha = \frac{3}{2\pi\mu_0}\int_r^1 \frac{\theta'(\xi, a)\,d\xi}{(\xi^2 - r^2)^{1/2}}.$$

If we write

$$\int_r^1 \frac{\theta'(\xi)\,d\xi}{(\xi^2 - r^2)^{1/2}} = (1 - r^2)^{1/2}(f_1 - if_2),$$

then the axial displacement assumes the form

$$u_{z0} = 3(1 - r^2)^{1/2}(2\pi\mu_0)^{-1}[p_s + p_0 \exp(-i\omega t)](f_1 - if_2).$$

Separating out the real part, we obtain

$$\mathrm{Re}\,u_{z0} = 3(1 - r^2)^{1/2}(2\pi\mu_0)^{-1}[p_s + p_0\Omega\cos(\omega t - \omega_0)], \tag{2.11}$$

where

$$\Omega = \sqrt{f_1^2 + f_2^2};\quad \omega_0 = \mathrm{arctg}\frac{f_2}{f_1}; \tag{2.12}$$

$$\begin{aligned} f_1 = 1 &- \frac{4 - r^2}{9\pi}k^2\sum_{n=0}^{\infty}\frac{E_{1n}}{n!}\beta^n - \frac{k^4}{75\pi}\Big[-\frac{41}{3\pi}E_{10}^2 + 68E_{30} + \\ &+ 4r^2\left(\frac{E_{10}^2}{\pi} + 6E_{30}\right) - r^4\left(\frac{E_{10}^2}{3\pi} + 2E_{30}\right)\Big] + \frac{k^4\beta}{75\pi}\Big[-\frac{41}{3\pi}E_{10}E_{11} + 68E_{31} + \\ &+ 4r^2\left(\frac{E_{10}E_{11}}{\pi} + 6E_{31}\right) - \frac{r^4}{\pi}\left(\frac{E_{10}E_{11}}{3} + 2\pi E_{31}\right)\Big] + \cdots; \end{aligned} \tag{2.13}$$

$$\begin{aligned} f_2 = \frac{2k^3}{3\pi}&\sum_{n=0}^{\infty}\frac{E^{2n}}{n!}\beta^n + k^5\Big[\frac{8}{45\pi}\left(7E_{40} - \frac{22}{3\pi}E_{10}E_{20}\right) + \frac{4r^2}{9\pi}\times \\ &\times\left(2E_{40} + \frac{E_{10}E_{20}}{6\pi}\right)\Big] + k^5\beta\Big[\frac{8}{45\pi}\left(7E_{41} - \frac{22}{3\pi}(E_{11}E_{20} + E_{10}E_{21})\right) + \\ &+ 2r^2(9\pi)^{-1}\left(4E_{41} + \frac{E_{11}E_{20} + E_{10}E_{21}}{3\pi}\right)\Big] + \cdots \end{aligned} \tag{2.14}$$

Thus, the displacement in the plane of the slit is given by formula (2.11). The terms in (2.13) and (2.14) containing the factor β take account of the effect of the inhomogeneity of the medium.

Representing the initial phase of the vibrations ω_0 in the form of the sum

$$\omega_0 = \omega_{01} + \omega_{02},$$

where $\omega_{01} = \mathrm{arctg}\,[(f_1/f_2)_{\beta=0}]$, from (2.12) we obtain a formula for the approximate determination of the perturbation of the initial phase of the vibrations under the effect of the inhomogeneity of the medium with low-frequency vibrations:

$$\omega_{02} = \mathrm{arctg}\frac{f_2 f_{10} - f_{20} f_1}{f_1 f_{10} + f_2 f_{20}}, \tag{2.15}$$

where f_{10} and f_{20} are equal, respectively, to f_1 and f_2 for $\beta = 0$. From (2.15) it follows that, under the above conditions, in a medium with the properties (1.1), the perturbation of the initial phase of the vibrations does not exceed the angle $\pi/4$.

In [7] a formula was obtained for determining the transverse cross section of an obstacle, scattering elastic waves in a solid medium. The transverse cross section S is defined by the ratio

$$S = W_1/W_0.$$

where W_0 is the energy of the incident wave, per unit area of the obstacle, normal to the direction of propagation of the wave; W_1 is the energy of the scattered wave, per unit area of a sphere of large radius R, surrounding the obstacle. In the case of a longitudinal wave, propagating parallel to the Ox axis,

$$S = 4\pi\frac{k}{\sqrt{3}}\,\mathrm{Im}\,g(0), \tag{2.16}$$

where g(0) is expressed in terms of the amplitude of the longitudinal scattered wave at a distant point of the field. Our aim is to derive a formula for the transverse cross section of a circular slit in the inhomogeneous medium under consideration. For this purpose, we write the expression for the axial displacement (2.5) at r = 0 somewhat differently:

$$u_z = \int_0^\infty \left[-\delta e^{-\delta |z|} + \frac{\alpha^2 (2\delta + a) e^{-\eta |z|}}{2\alpha^2 - k^2 + a\eta} \right] A_1(\alpha) \alpha d\alpha. \tag{2.17}$$

We rewrite the first term in the form

$$3\sqrt{3} N_1 = -k^3 i \int_0^1 A_1\left(k \sqrt{\frac{1-v^2}{3}}\right) e^{i\frac{kv|z|}{\sqrt{3}}} v^2 dv + k^3 \int_0^\infty A_1\left(k \sqrt{\frac{1-u^2}{3}}\right) e^{\frac{-ku|z|}{\sqrt{3}}} \times$$
$$\times u^2 du.$$

where $v = \sqrt{1-\zeta^2}$; $\zeta = \alpha\sqrt{3}/k$; $u = \sqrt{\zeta^2 - 1}$.

As $z \to \infty$, we obtain an asymptotic representation for $N_1(k, z)$ [8]:

$$N_1 = -A_1(0) \frac{k^2 \exp\left(\frac{ik|z|}{\sqrt{3}}\right)}{3|z|}. \tag{2.18}$$

We transform the second term in (2.17) analogously:

$$N_2 = -k \int_0^1 A_1(k\sqrt{1-v^2}) e^{-ivk|z|} G(v) v (1-v^2) dv + k \int_0^\infty A_1(k\sqrt{1+u^2}) \times$$
$$\times e^{-ku|z|} G(u) u (1+u^2) du,$$

where

$$G(\alpha) = \frac{2\delta + a}{2\alpha^2 - k^2 + a\eta}; \quad \alpha = k\zeta; \quad \zeta = \begin{cases} \sqrt{1-v^2}, & \zeta < 1, \\ \sqrt{1+u^2}, & \zeta > 1. \end{cases}$$

Again using the results of [8], we find that, as $z \to \infty$, $N_2 = 0$. Then from (2.18) we find that $g(0) = -(k^2/3)A_1(0)$, and (2.16) is brought to the form

$$S = -\frac{4\pi k}{\sqrt{3}} \operatorname{Im} A_1(0).$$

Starting from (1.8), (1.9), and (1.15), we obtain

$$A_1(0) = \lim_{\alpha \to 0} \frac{2\alpha^2 - k^2 + a\eta}{\alpha\pi [\delta k^2 + a(\alpha^2 - \eta\delta)]} \int_0^1 \theta(\xi) \sin(\alpha\xi) d\xi = \frac{2\sqrt{3}i}{k\pi} \int_0^1 \xi\theta(\xi) d\xi.$$

Thus, we finally obtain

$$S = -8 \operatorname{Re} \int_0^1 \xi\theta(\xi) d\xi.$$

Substituting here the values of $\theta(\xi)$ from (1.23) and limiting ourselves to a few terms, we express the transverse cross section as a function of the two parameters k and β:

$$S = 8 \left\{ 1 + \frac{2k^2}{5\pi}\left(-E_{20} + \beta E_{21} - \frac{\beta^2}{2} E_{22} + \ldots \right) + \frac{k^4}{105\pi^2} \left[17E_{20}^2 + 108E_{40} - \right.\right.$$
$$- \beta(34E_{21}E_{20} + 108\pi E_{41}) + \beta^2(17E_{22}E_{20} + 54\pi E_{42}) - \beta^3(17E_{22}E_{21} +$$
$$\left.\left. + 18\pi E_{43}) + \frac{\beta^4}{2}(17E_{22}^2 + 18\pi E_{44}) + \ldots \right] + \ldots \right\}.$$

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DEFORMATION OF A STOCHASTICALLY INHOMOGENEOUS SOLID WITH AN OPENING

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At the present time, there exist a very large number of solutions of the deformation of unbounded stochastically inhomogeneous bodies. In these solutions effective moduli of elasticity are determined, i.e., the mechanical properties of the material, averaged over the spatial region, as a function of the parameters characterizing the structural inhomogeneity of the medium. However, by virtue of the nonlocal character of the connection between the mean stresses and the mean deformations [1], the effective moduli of elasticity depend to a considerable degree on the boundary-value problem. It must be noted that, for the solution of concrete boundary-value problems in the theory of elasticity, considerable mathematical difficulties arise. At the present time, there exist solutions to a number of boundary-value problems of the stochastically inhomogeneous theory of elasticity for a half-plane, a band [1], and an infinite plane with a circular opening [2, 3]. In the case of antiplane deformation, a solution has been given to the problem of the propagation of a crack in a stochastically inhomogeneous body [4, 5].

We consider the plane problem of the deformation of a body, whose elastic constants are random functions of the coordinates. With the surface forces g_i given at the contour L of the region S, occupied by the body, and in the absence of volumetric forces, the equations of the plane problem of the theory of the elasticity of isotropic inhomogeneous bodies, written in terms of the stresses, have the form [1]

$$\begin{gathered} X_n = \sigma_x \cos nx + \tau_{xy} \cos ny = g_1, \\ y_n = \tau_{xy} \cos nx + \sigma_y \cos ny = g_2, \\ \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0, \ \nabla^2 [\gamma(\sigma_x - \sigma_y)] = \frac{\partial^2 q}{\partial x^2}\sigma_x + 2\frac{\partial^2 q}{\partial x \partial y}\tau_{xy} + \frac{\partial^2 q}{\partial y^2}\sigma_y, \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0. \end{gathered} \tag{1}$$

Here q and γ are expressed in terms of the Young modulus E(x, y) and the Poisson coefficient ν(x, y) by the relationships

$$\gamma = 1/E, \ q = (1 + \nu)/E, \tag{2}$$

where ∇^2 is a Laplace operator.

Let q(x, y) and γ(x, y) be random functions of the coordinates. Then relationships (1) constitute a stochastically nonlinear problem, determining the random functions τ_{ij}(x, y). We represent the values of q and γ in the form $q = \langle q \rangle + q'$, $\gamma = \langle \gamma \rangle + \gamma'$. We postulate that the random functions q(x, y) and γ(x, y) are statistically homogeneous and are statistically homogeneously interconnected ($\langle q \rangle$ = const, $\langle \gamma \rangle$ = const). If the solution of the problem (2) is represented in the form of a series in powers of some parameter $\varkappa$, then the problem (1) is normalized. The parameter $\varkappa$ is introduced by the relationships [1].

$$q = \langle q \rangle + \varkappa q', \ \gamma = \langle \gamma \rangle + \varkappa \gamma', \ \tau_{ij} = \sum_{k=0}^{\infty} \varkappa^k \tau_{ij}^{(k)}. \tag{3}$$

Substituting (3) into (1), and equating expressions with identical powers of $\varkappa$, we obtain a boundary-value problem for the zero approximation,

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